JOURNAL OF DIFFERENTIAL EQUATIONS 40, 155–167 (1981)

Spectral Theory for Linearizations of Dynamical Systems

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Received January 14, 1980; revised June 23, 1980

1. INTRODUCTION

The description of the asymptotic growth rates for the solution trajectories of a system of differential equations is a central problem in the qualitative theory of dynamical systems. These growth rates are described by the eigenvalues of the Jacobian matrix for trajectories near a critical point, by the Floquet exponents for trajectories close to a periodic orbit and, generally, by the Lyapunov characteristic exponents in a neighborhood of an arbitrary trajectory. In the examples, the asymptotic behavior of the solution trajectories is given by the association of a "spectrum," and this is a recurrent theme in the method of linearization.

Sacker and Sell [10] recently introduced a spectral theory for differential systems of quite general type: the (linear) skew-product flows (see Section 3). Their work is in the spirit of the classical linearization theory and is closely tied to the idea of the Lyapunov exponents. We prefer to use the word "spectrum" literally, identifying it with the spectrum of an operator on a Banach space. In general, we consider a one-parameter group ϕ^t of homeomorphisms of a compact metric space M which, for example, may be the solution flow of a differential equation on a smooth manifold. Second, we assume that a continuous vector bundle (E, M) is defined, i.e., a locally trivial continuous map $\pi: E \to M$ with linear fibers which are isomorphic to either real n-dimensional space $\mathbf{R}^n$ or complex n-dimensional space $\mathbf{C}^n$. In case ϕ^t is the flow of a vector field X on M, this vector bundle could be the tangent bundle TM. Finally, we assume that there is group of homeomorphisms Φ^t on E with $\pi\Phi^t = \phi^t\pi$ and such that $\Phi^t_x: E_x \to E_{\phi t(x)}$ is a linear isomorphism for each $x \in M$. For instance, in case ϕ^t is smooth, Φ^t could be the tangent flow on TM given by the derivative of ϕ^t. Using the compactness of M we define a Finsler on E which continuously assigns a norm, denoted $|\ |$, to each fiber E_x. With this assignment, the space $\Gamma(E)$ of continuous sections of E (functions $\eta: M \to E$ such that $\pi \circ \eta(x) = x$) forms a Banach space with norm $\|\eta\| = \sup_{x\in M} |\eta(x)|$. The topology of $\Gamma(E)$ is

0022-0396/81/050155–13$02.00/0

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independent of the choice of Finsler. The vector bundle flow (Φ^t, ϕ^t) induces a strongly continuous group of bounded linear operators $\Phi_t^{\#}$ on $\Gamma(E)$ defined for $\eta \in \Gamma(E)$, as follows:

$$\Phi_t^{\#}\eta = \Phi^{-t} \circ \eta \circ \phi^t.$$

In these definitions one may take $t \in \mathbf{R}$ or, in the discrete case, $t \in Z$. When $t \in \mathbf{R}$, we also define the infinitesimal generator L of $\Phi_t^{\#}$ for $\eta \in \Gamma(E)$ by setting

$$L\eta = \frac{d}{dt} \Phi_t^{\#}\eta \bigg|_{t=0}.$$

Of course, in this context the "spectrum" is either the spectrum $\sigma(\Phi_t^{\#})$ of $\Phi_t^{\#}$ or the spectrum $\sigma(L)$ of L.

This point of view goes back at least to the work of Mather [7]. Although important results have been obtained by many other authors, we cite the contributions of Mañe [8, 9] and Hirsch, Pugh and Shub [5] which are especially important for our work. These authors study the discrete vector bundle flow (Tf, f) defined in the tangent bundle TM by the derivative of a diffeomorphism f on a compact manifold. Recall that f is called Anosov if TM splits as an invariant continuous Whitney sum $TM = E^+ \oplus E^-$ and Tf exponentially contracts E^+ for positive iterates and exponentially contracts E^- for negative iterates. Also, define as above, for $\eta \in \Gamma(TM)$

$$f^{\#}\eta = Tf^{-1} \circ \eta \circ f.$$

Mather's fundamental insight is the theorem: f is Anosov if and only if $f^{\#}$ is a hyperbolic operator on $\Gamma(TM)$, i.e., $\sigma(f^{\#})$ contains no complex number with modulus one. When f^t defines a flow, we have proved [1] the analogue of Mather's theorem: f^t is Anosov if and only if $f_t^{\#}$, for each $t > 0$, is a hyperbolic operator on the space of sections $\Gamma(TM/[X])$ where $X = df^t/dt$. Also, in the flow case, we have proved [1] the theorem: f^t is Anosov if and only if the generator L of $f_t^{\#}$ is infinitesimally hyperbolic on $\Gamma(TM/[X])$, i.e., $\sigma(L)$ contains no complex number with real part zero. Analogues of both of these results are valid [2] for smooth vector bundle flows (Φ^t, ϕ^t), and in case ϕ^t preserves a volume, both results are valid [1] when the spectra are computed in $\Gamma^2(E)$, the Lebesgue space of square integrable sections.

In the present article we extend the theory to include the definitive Annular Hull Theorem of Section 2, with the remaining sections devoted to the applications. In Section 3 we relate, in a precise way, our spectral results to the spectrum defined by Sacker and Sell, incidentally obtaining many of their results. In addition, our approach allows us to completely resolve the question of when the spectrum of a rank 1 vector bundle flow is "discrete"

(Theorem 3.9). Sections 4 and 5 treat, respectively, the problems of detecting vector fields with trivial centralizers and the problem of finding unintegrated criteria for a flow to be normally hyperbolic at an invariant submanifold.

After this paper was written Russell Johnson sent us a preprint [13] in which he also relates the Sacker–Sell spectrum to the spectrum of the operator $\Phi^{\#}$. His result is the same as ours in the special case he considers.

2. The Annular Hull Theorem

As above, we shall assume the existence of a continuous vector bundle flow (Φ^t, ϕ^t) acting on the continuous vector bundle (E, M) with base space M a compact metric space. Let $L = L_\Phi$ denote the infinitesimal generator of the associated c_0-semigroup $\Phi_t^{\#}$ acting on the Banach space of continuous sections $\Gamma(E)$. Given any set A of complex numbers, define the *annular hull of* A, written $\mathscr{A}(A)$ to consist of the totality of circles centered at the origin which meet the set A. Our main result in this section is the following spectral theorem:

THEOREM 2.1. (a) $\exp(t\sigma(L)) \subseteq \sigma(\Phi_t^{\#}) \subseteq \mathscr{A}(\exp(t\sigma(L)))$.

(b) [1, Theorem 2.1]. *Suppose* (Φ^t, ϕ^t) *is a smooth vector bundle flow. Then, if the nonperiodic points of* ϕ^t *are dense,*

$$\exp(t\sigma(L)) = \sigma(\Phi_t^{\#}).$$

We shall derive this result from the next proposition.

PROPOSITION 2.2. *L is infinitesimally hyperbolic if and only if $\Phi_t^{\#}$ is hyperbolic for some (hence, all) $t \neq 0$.*

In [2] we proved (2.2) when (Φ^t, ϕ^t) is a smooth vector bundle flow acting on a smooth vector bundle. Since the primary objective of this article is to give applications of (2.1) and since the proof of (2.2) in the present setting is similar to the smooth case, we shall only provide an outline of the proof of (2.2).

A sketch of the proof of (2.2): Let $i\mathbf{R}$ denote the imaginary complex numbers, and let S^1 denote the unit circle. We must show that $\sigma(L) \cap i\mathbf{R} = \varnothing$ if and only if $\sigma(\Phi_t^{\#}) \cap S^1 = \varnothing$. The inclusion

$$\exp(t\sigma(L)) \subseteq \sigma(\Phi_t^{\#})$$

holds for any c_0-semigroup (see [4], for relevant semigroup theory). Hence, hyperbolicity implies infinitesimal hyperbolicity. Also, for abstract c_0-semigroups the point and residual spectra of L exponentiate to give the corresponding point and residual spectra of $\Phi_t^{\#}$. Therefore, the result will follow if $\sigma_{\mathrm{ap}}(\Phi_t^{\#}) \cap S^1 \neq \varnothing$ implies $\sigma_{\mathrm{ap}}(L) \cap i\mathbf{R} \neq \varnothing$, where σ_{ap} denotes the approximate point spectrum.

One can easily adapt a result of Mañe (cf. [8, p. 367]) for tangent flows ($\Phi^t = T\phi^t$) to conclude that $\sigma_{\mathrm{ap}}(\Phi_t^{\#}) \cap S^1 \neq \varnothing$ if and only if there is a nonzero vector $v \in E$ and

$$\sup\{|\Phi^t v| : t \in R\} < \infty. \tag{1}$$

In effect, the proof of (2.2) reduces to showing that (1) holds only if $\sigma(L) \cap iR \neq \varnothing$. To accomplish this, we decompose M into the sets BP and $M \backslash BP$ where BP is the set of all periodic orbits which admit a tubular neighborhood consisting entirely of periodic orbits with uniformly bounded prime period. If (1) holds, for v in the fiber E_x over $x \in BP$, it follows that the infinitesimal generator l of $\Phi_t^{\#}$, acting on sections of E along the orbit $\mathcal{O}_x$, has an eigenvalue $i\alpha$, $\alpha \in \mathbf{R}$. Given $\varepsilon > 0$, the corresponding "eigensection" ξ of $i\alpha$ can be extended off $\mathcal{O}_x$ to a section η of unit length such that

$$\|L\eta - i\alpha\eta\| \leqslant \varepsilon.$$

Therefore $i\alpha \in \sigma(L)$.

On the other hand, suppose (1) holds for $v \neq 0$ in E_x and $x \in M \backslash BP$. Then x is either nonperiodic or every neighborhood of x contains points of arbitrarily large prime period. Condition (1) implies that $\sigma(\Phi_t^{\#}) \cap S^1 \neq \varnothing$ for all $t \neq 0$. In particular, we may suppose that for $\varepsilon > 0$, there exists $\eta \in \Gamma(E)$ such that $\|\eta\| = 1$, $\|e^{-i\alpha}\Phi_1^{\#}\eta - \eta\| < \varepsilon$, and $|\eta(x)| = 1$. The construction detailed in [1, Theorem 2.1] now implies that $i\alpha \in \sigma_{\mathrm{ap}}(L)$, which completes the sketch of the proof. Q.E.D.

Our next task is to derive (2.1) from (2.2).

Proof of (2.2). As remarked in the proof of (2.1), the inclusion $\exp(t\sigma(L)) \subseteq \sigma(\Phi_t^{\#})$ already follows from the spectral theory for c_0-semigroups. To establish the second inclusion, we consider the case $t = 1$, and suppose that $e^{\lambda} \notin A$, where A denotes the annular hull of $\exp(\sigma(L))$. It follows that $e^{\lambda + ir} \notin A$ for all $r \in \mathbf{R}$. By definition of A, $[L - (\lambda + ir)]^{-1}$ exists for all $r \in \mathbf{R}$. Define a new semigroup $\Psi_t^{\#} = e^{-\lambda t}\Phi_t^{\#}$ with generator $K = L - \lambda$. Then K is infinitesimally hyperbolic and, hence, from (2.1) it follows that $\Psi^{\#}$ is hyperbolic, i.e., $(\Phi^{\#} - e^{\lambda + ir})^{-1}$ exists for all $r \in \mathbf{R}$. Q.E.D.

EXAMPLE 2.3. To show that it is really necessary to invoke the annular hull, consider the following example: Let $E = S^1 \times \mathbf{R}$ denote the trivial bundle over S^1. Then, $\Gamma(E)$ consists of the continuous maps from S^1 to $\mathbf{R}$. Define the vector bundle flow $(\Phi^t, \phi^t) = (Id, T_t)$ where $T_t(\theta) = \theta + t$ and Id

denotes the identity map of $\mathbf{R}$. Using Fourier series, it is not hard to show that $\sigma(L) = \{in: n \in Z\}$, whereas $\sigma(\Phi_t^{\#}) = S^1$ for $t \neq 0$.

Remark. As an immediate consequence of the Annular Hull Theorem and the example we can conclude that even though the spectrum of L does not exponentiate to equal the spectrum of $\Phi^{\#}$, the infinitesimal hyperbolicity of L does exponentiate to give the hyperbolicity of (Φ^t, ϕ^t). Thus, a vector bundle flow (Φ^t, ϕ^t) in a vector bundle over a compact metric space is hyperbolic (admits an invariant splitting with exponential growth and decay) if and only if the infinitesimal generator L is infinitesimally hyperbolic.

3. The Spectrum of a Linear Skew-Product Flow

In [10] Sacker and Sell introduced a "spectrum" for linear skew-product flows. Let (Φ^t, ϕ^t) denote, as before, a vector bundle flow on (E, M) where $t \in \mathbf{R}$ or Z and where M is a compact metric space. For $\gamma \in \mathbf{R}$ define the vector bundle flow $\pi_\lambda = (e^{-\lambda t}\Phi^t, \phi^t)$ and, for notational convenience, set $\Phi_\lambda^t = e^{-\lambda t}\Phi^t$. For $N \subseteq M$, Π_λ admits an exponential dichotomy over N if there is a continuous projector $P: E|_N \to E|_N$ and positive constants K and α such that

$$|\Phi_\lambda^t P(y)\, \Phi_\lambda^{-s}| \leqslant K e^{-\alpha(t-s)}, \qquad s \leqslant t,$$

and

$$|\Phi_\lambda^t (I - P(y))\, \Phi_\lambda^{-s}| \leqslant K e^{-\alpha(s-t)}, \qquad t \leqslant s$$

for all $y \in N$.

For $y \in M$, define the resolvent $\rho(y)$ [10, p. 327] by

$$\rho(y) = \{\lambda \in R: \pi_\lambda \text{ admits an exponential dichotomy over } y\}.$$

The spectrum $\Sigma(y)$ is defined by

$$\Sigma(y) = R - \rho(y).$$

For a subset $N \subseteq M$ the spectrum $\Sigma(N)$ is defined by

$$\Sigma(N) = \bigcup_{y \in N} \Sigma(y)$$

and the resolvent $\rho(M)$ is defined by

$$\rho(N) = \bigcap_{y \in N} \rho(y) = \mathbf{R} - \Sigma(N).$$

In case N is a compact invariant set in M one has

LEMMA 3.1 [10, p. 327]. *$\lambda \in \rho(N)$ if and only if π_λ admits an exponential dichotomy over N.*

We consider the vector bundle $(E|_N, N)$ and define $\Phi^{\#}$ and L in the space $\Gamma(E|_N)$. Using these operators we can identify the spectrum $\Sigma(N)$.

THEOREM 3.2. (a) *In the flow case, $\lambda \in \Sigma(N)$ if and only if $\lambda + i\alpha \in \sigma(L)$ for some $\alpha \in \mathbf{R}$.*

(b) *$\lambda \in \Sigma(N)$ if and only if $e^{\lambda + i\alpha} \in \sigma(\Phi^{\#})$ for some $\alpha \in \mathbf{R}$.*

(c) *Assume that (Φ^t, ϕ^t) is a smooth vector bundle flow and that the nonperiodic points of ϕ^t are dense. Then, $\lambda \in \Sigma(N)$ if and only if $e^{\lambda + i\alpha} \in \sigma(\Phi^{\#})$ for all $\alpha \in \mathbf{R}$.*

Proof. In view of the Annular Hull Theorem, (a) follows from (b). To show (b) note that $\lambda \in \rho(N)$ if and only if π_λ admits an exponential dichotomy over N. But, this is true if and only if the vector bundle flow π_λ is hyperbolic on $E|_N$, i.e., $e^{i\alpha} \in \rho(e^{-\lambda}\Phi^{\#})$ for all $\alpha \in \mathbf{R}$. But, π_λ is hyperbolic if and only if $\Phi^{\#} - e^{\lambda + i\alpha}$ is invertible for all $\alpha \in \mathbf{R}$. Part (c) follows from Theorem 2.1(b). Q.E.D.

Since Σ is a true spectrum, up to imaginary translation, we obtain from Theorem 3.2

COROLLARY 3.3. (a) [10, Lemma 6] *$\Sigma(N)$ is compact.*

(b) [10, Lemma 7] *$\Sigma(N)$ is nonempty.*

Also, recall the spectral Decomposition Theorem [10, Theorem 2].

THEOREM 3.4. *Assume $\dim E = n$ and N is an invariantly connected compact invariant set for (Φ^t, ϕ^t). Then, $\Sigma(M) = [a_1, b_1] \cup \cdots \cup [a_k, b_k]$, the union is disjoint and E splits as a continuous Whitney sum*

$$E = V_1 \oplus V_2 \oplus \cdots \oplus V_k$$

with the spectrum $\Sigma_i(N)$ of the restriction of (Φ^t, ϕ^t) to (V_i, N) given by $\Sigma_i(N) = [a_i, b_i]$.

For the case in which M is a manifold the last result follows from the Annular Hull Theorem and the Spectral Decomposition Theorem of Hirsch, Pugh, and Shub [5, Proposition 2.2]. The latter theorem extends readily to cover the compact metric space case.

The main motivation for studying $\Sigma(N)$ comes from the theory of almost periodic differential equations. Let $\mathscr{A}$ denote the set of all continuous $(n \times n)$-matrix-valued functions $A(s)$ which are almost periodic in $s \in \mathbf{R}$ and give $\mathscr{A}$ the topology generated by the sup norm $\|A\|_\infty$. Also, define the translation operator ϕ^t on $\mathscr{A}$ by

$$\phi^t A(s) = A(t+s).$$

Then, the hull of A, $H(A)$, is the closure of the ϕ^t orbit of A. Let F denote either $\mathbf{R}^n$ or $\mathbf{C}^n$ and define for each $t \in R$ the linear operator Φ^t on F such that $\Phi^t x$, for $x \in F$, is the solution vector at time t of the initial value problem

$$x' = A(s)x, \qquad x(0) = x.$$

Clearly, (Φ^t, ϕ^t) defines a vector bundle flow (linear skew-product flow) on $H(A) \times F$.

The central problem in the study of almost periodic systems, from this point of view, is to determine when the spectrum $\Sigma(H(A))$ is discrete, i.e., consists of a finite number of points. Two facts are known: (a) if fiber $\dim F = 1$, then $\Sigma(H(A))$ is discrete [10, Theorem 7] and (b) If $\dim F \geqslant 2$, then, in general, $\Sigma(H(A))$ is not discrete. The first statement has been generalized [10, p. 353] to arbitrary vector bundle flows when $\dim F = 1$ and M is a compact minimal set with unique invariant measure (= uniquely ergodic).

We now show that when $\dim F = 1$ and M is a compact Hausdorff space, it is possible to compute the spectral radius in terms of the invariant measures. To this end, let M denote a compact Hausdorff space and let $C(M)$ denote the Banach space of continuous complex-valued functions on M. If $h \in C(M)$ and ϕ is a continuous transformation of M, define the operator T on $C(M)$ by

$$Tf = hUf = h(f \circ \phi).$$

Also, recall that the spectral radius $r(T)$ of T is given by

$$r(T) = \sup |\sigma(T)|.$$

Let $\mathscr{B}_\phi$ denote the set of ϕ-invariant Borel measures on M.

THEOREM 3.5. $r(T) = \sup\{\exp \int_M \log |h| \, d\mu : \mu \in \mathscr{B}_\phi\}$.

We thank our colleague Jim Roberts [12] for the method, developed below, for constructing invariant measures. See also [14, pp. 486–520].

Proof. We have the identities

$$\|T^n\| = \left\| \left(\prod_{k=0}^{n-1} U^k h \right) U^n \right\| = \left\| \prod_{k=0}^{n-1} U^k h \right\|_\infty .$$

Also, the spectral radius formula

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$$

implies

$$\log(r(T)) = \lim_{n\to\infty} \frac{1}{n} \log \left\| \prod_{k=0}^{n-1} U^k h \right\|_\infty.$$

Choose a sequence $\{x_n\} \subset M$ such that

$$\left| \left(\prod_{k=0}^{n-1} U^k h \right) (x_n) \right| = \left\| \prod_{k=0}^{n-1} U^k h \right\|_\infty$$

and define δ: $C(M) \to l^\infty$ by

$$\delta(g)(n) = \frac{1}{n} \sum_{k=0}^{n-1} (U^k g)(x_n).$$

We now construct a ϕ-invariant Borel measure μ such that

$$r(T) = \exp \int_M \log |h| \, d\mu.$$

Let c denote the space of convergent sequences and let $\lim: c \to \mathbf{R}$ be the functional which assigns to each sequence its limit. Evidently, $\|\lim\| = 1$ and, by the Hahn–Banach theorem, there is a linear functional Λ on l^∞ which extends lim, such that $\|\Lambda\| = 1$. Now, define μ: $C(M) \to \mathbf{R}$ by

$$\mu(g) = \Lambda\delta(g).$$

One verifies easily that μ is a positive linear functional with $\mu(1) = 1$ and $\|\mu\| = 1$. Also, it follows that μ is ϕ-invariant and since $\|\mu\| = 1$ that $\mu(M) = 1$ when we regard μ as a Borel measure. Since

$$\exp \int_M \log |h| \, d\mu = \exp(\mu(\log |h|)) = r(T),$$

we have

$$r(T) \leqslant \sup \left\{ \exp \int_M \log |h| \, d\mu \colon \mu \in \mathscr{B}_\phi \right\}.$$

Moreover, if $\mu \in B_\phi$ and $n \in Z^+$ we have

$$\exp \int_M \log |h| \, d\mu = \exp \frac{1}{n} \int_M \log \left| \prod_{k=0}^{n-1} U^k h \right| d\mu$$
$$\leqslant \exp \left(\log \int_M \left| \prod_{k=0}^{n-1} U^k h \right|^{1/n} d\mu \right)$$
$$\leqslant \left\| \prod_{k=0}^{n-1} U^k h \right\|_\infty^{1/n}.$$

Hence,

$$\sup \left\{ \exp \int_M \log |h| \, d\mu : \mu \in B_\phi \right\} \leqslant r(T).$$

Q.E.D.

Corollary 3.6. *$\sigma(T)$ is contained in the annulus in the complex plane with radii r_i, $i = 1, 2$, where*

$$r_1 = \sup \left\{ \exp \int_M \log |h| \, d\mu : \mu \in B_\phi \right\}$$

and

$$1/r_2 = \sup \left\{ \exp \int_M - \log |h| \, d\mu : \mu \in B_\phi \right\}.$$

Proof. As $T^{-1} = (1/U^{-1}h)\, U^{-1}$, it follows that

$$r(T^{-1}) = 1/r_2.$$

But, if $\lambda \in \sigma(T)$, then λ lies in the annulus with radii $r(T)$ and $1/r(T^{-1})$.

Q.E.D.

If (Φ^t, ϕ^t) is a vector bundle flow on $(M \times F, M)$ where $F = R$ or $\mathbf{C}$, it follows immediately that $\Phi^{\#}$ acting on $\Gamma(M \times F)$ may be identified with the operator $T = hU$ acting on $C(M)$ for some function $h \in C(M)$.

Corollary 3.7. *If ϕ has a unique invariant Borel measure on M and M is a compact Hausdorff space, then $\sigma(\Phi^{\#})$ is a circle in the complex plane and, hence, $\Sigma(M)$ is discrete.*

Proof. It follows from [7] that $\sigma(\Phi^{\#})$ is rotationally invariant. But, since there is a unique invariant measure, Corollary 3.6 implies that $r_1 = r_2$, i.e., $\sigma(T)$ is contained in a circle in the complex plane. Q.E.D.

Remark. If $A(s) \in \mathscr{A}$, and $\phi A(s) = A(s+1)$, then ϕ has a unique invariant measure on $H(A)$. Hence, for almost periodic differential equations if $\dim F = 1$ the flow on $H(A) \times F$ has discrete spectrum.

We now want to show that the general (rank 1) vector bundle case reduces to the preceding trivial line bundle case by a canonical procedure. Let (Φ, ϕ) be a vector bundle automorphism on the continuous bundle (E, M) whose fibers are either copies of $\mathbf{C}$ or $\mathbf{R}$. (In the complex case Φ must correspond to multiplication by a complex scalar; otherwise, Φ is not $\mathbf{C}$-linear.)

Choose a continuous Finsler $\| \ \|$ on E. We shall define an associated vector bundle map (Ψ, ϕ) on the trivial bundle $M \times \mathbf{K}$ with $\mathbf{K}$ either $\mathbf{C}$ or $\mathbf{R}$, depending on the fiber choice for E. Let $\Phi_x: E_x \to E_\phi(x)$ be the associated fiber transformation, with operator norm $\|\Phi_x\|$. Define Ψ by setting $\Psi(x, r) = (\phi(x), \|\Phi_x\| r)$. With this choice $\Psi^\# = hU_\phi$, where $h(x) = \|\Phi_x\|$ for all x in M. It is easy to prove the following:

PROPOSITION 3.8. *The maps $\Phi^\#$ and $\Psi^\#$ have the same norm and spectral radius.*

It is clear that Proposition 3.8 yields a canonical procedure for calculation the annular hull of $\sigma(\Phi^\#)$. Namely, associate the operator $\Psi^\#$ with $\Phi^\#$ as above and $\Delta^\#$ with $(\Phi^{-1})^\#$ in accord with the above procedure. Then $\Psi^\# = hU_\phi$, $\Delta^\# = gU_\phi$, where $h(x) = \|\Phi_x\|$ and $g(x) = \|\Phi_x^{-1}\| = \|\Phi_{\phi^{-1}x}\|^{-1}$, since the transformations Φ_x are 1-dimensional. But this implies $\Delta = \Psi^{-1}$. Combining these remarks with Corollary 3.6 we obtain the fact that $\sigma(\Phi^\#)$ and $\sigma(\Psi^\#)$ have the same annular hull. We rephrase these observations as the final theorem of this section.

THEOREM 3.9. *Let (Φ, ϕ) be a vector bundle automorphism on a continuous rank* 1 *bundle (E, M), with associated operator $\Phi^\#$ on the Banach section space $\Gamma(E)$. Let r_i, $i = 1, 2$ be as in Corollary* 3.6 *with $h(x) = \|\Phi_x\|$ for all $x \in M$. Then the following hold*:

(1) *The annular hull of $\sigma(\Phi^\#)$ is the annulus with radii r_1 and r_2.*

(2) *The spectrum $\Sigma(M)$ is discrete if and only if the integral*

$$\int_M \log \|\Phi_x\| \, d\mu$$

has the same value for all invariant measures μ.

4. CENTRALIZERS OF ANOSOV FLOWS

As in [6] we define the centralizer, $\mathscr{C}(X)$, of a vector field X by

$$\mathscr{C}(X) = \{Y \mid [X, Y] = 0\},$$

where [,] denotes the Lie bracket; $\mathscr{C}(X)$ is called trivial if every $Y \in \mathscr{C}(X)$ is a function multiple of X.

THEOREM 4.1 [6, 11]. *If X is Anosov, then $\mathscr{C}(X)$ is trivial.*

Using the Annular Hull Theorem we can prove a much stronger version of Theorem 4.1. To state our theorem we let ϕ^t be the flow of X on the compact manifold M and let $\Phi^t = T\phi^t$ denote the induced tangent flow on the tangent bundle TM. The tangent bundle flow (Φ^t, ϕ^t) induces a vector bundle flow on the quotient $E = TM/[X]$. Moreover, the induced group $\Phi_t^\#$ on $\Gamma(E)$ has the infinitesimal generator L which is induced by the Lie derivative operator L_X. As a consequence of Proposition 2.2 we have

THEOREM 4.2. *X is Anosov if and only if L is infinitesimally hyperbolic. In particular, if $[X, Y] = fX$ where $f: M \to \mathbf{R}$, then $Y = gX$ for $g: M \to \mathbf{R}$.*

Proof. The characterization is proved in [1]. Note that if $L_X Y = fX$, then the section $[Y] \in \Gamma(E)$ induced by Y is in the kernel of L. But, this implies $[Y] = 0$, i.e., $Y = gX$ for some function g. Q.E.D.

Remark 1. Our proof holds for any Y which induces a section $[Y]$ in the domain of L; hence, we do not need to assume that $Y \in C^1$. Also, one does not need the full power of the Annular Hull Theorem to deduce from $[X, Y] = fX$ that $Y = gX$. We only need the fact that L is injective on $\Gamma(E)$. At present, we do not know how to characterize the flows ϕ^t such that the induced L is injective. But, for example, if ϕ^t is quasi-Anosov, i.e., Φ^t has no bounded orbits, then L is injective (cf. [8]) and X has a trivial centralizer.

Remark 2. It is interesting to note that if Y is a reparametrization of an Anosov vector field X, then Y is also Anosov. This standard result is an easy consequence of our theorem. In fact, if X is Anosov, L_X has a bounded inverse J on $\Gamma(E)$. Clearly, JM_f is the bounded inverse of L_{fX} where M_f is the operator defined by

$$M_f[Y] = \left[\frac{1}{f}\, Y\right].$$

5. Normally Hyperbolic Flows

In this section we shall make extensive use of the formulation of normal hyperbolicity given by Hirsch, Pugh and Shub [5]. We refer the reader to that reference for a thorough treatment.

Let ϕ^t be a smooth flow on a compact manifold M and set $X = d\phi^t/dt$. If $P \subseteq M$ is a smooth ϕ^t-invariant submanifold, i.e., $\phi^t P = P$ for all $t \in \mathbf{R}$, a strong version of the normal hyperbolicity of the flow at P called "immediate, absolute 1-normal hyperbolicity at P" asserts the following [5, p. 3]: the tangent bundle over P, T_PM, decomposes into a continuous $T\phi^t$-invariant vector bundle splitting

$$T_PM = TP \oplus N^sP \oplus N^uP,$$

such that for all $t \neq 0$, $k = 0$, 1, and $p \in P$

$$\text{(a)} \quad \inf_{p \in P} \| T\phi^{-t} \mid N_p^u P \|^{-1} > \sup_p \| T\phi^t \mid T_p P \|^k$$

and

$$\text{(b)} \quad \sup_p \| T\phi^t \mid N_p^s P \| < \inf_p \| T\phi^{-t} \mid T_p P \|^{-k}.$$

For convenience we shall say that $\{\phi^t\}$ is *normally hyperbolic at P* if (a) and (b) hold for at least one (and, thus, all) $t \neq 0$.

Then, consider the Banach section spaces $\Gamma(T_pM)$, $\Gamma_1 = \Gamma(TP)$, and the quotient Banach space $\Gamma_2 = \Gamma(T_pM)/\Gamma(TP)$. Let $\phi^{\#} = (T\phi^1)^{\#}$, and let $[\phi^{\#}]$ denote the induced map on Γ_2.

PROPOSITION 5.1 [5]. *The flow ϕ^t is normally hyperbolic at P if and only if the spectrum $\sigma(\phi^{\#}, \Gamma_1)$ lies in an annulus centered at zero and disjoint from the set $\sigma([\phi^{\#}], \Gamma_2)$.*

We shall prove the infinitesimal version of Proposition 5.1.

THEOREM 5.2. *The flow ϕ^t is normally hyperbolic at P if and only if $\sigma(L_X, \Gamma_1)$ lies in a vertical strip disjoint from the spectrum $\sigma(L_X, \Gamma_2)$.*

Proof. We claim Γ_2 is isomorphic, as a Banach space, to $\Gamma(T_pM/TP)$. To see this, notice that with the obvious maps the exactness of the vector bundle sequence

$$0 \to TP \to T_pM \to T_pM/TP \to 0$$

implies that the sequence

$$0 \to \Gamma(TP) \to \Gamma(T_pM) \to \Gamma(T_pM/TP) \to 0$$

is exact. Hence, there is a canonical isomorphism

$$F: \Gamma(T_pM)/\Gamma(TP) \to \Gamma(T_pM/TP).$$

As

$$\sigma([\phi^{\#}], \Gamma_2) = \sigma(F[\phi^{\#}]\, F^{-1}, \Gamma(T_pM/TP)),$$

the Annular Hull Theorem implies that $\sigma(L_X, \Gamma_1)$ lies in a vertical strip disjoint from the spectrum $\sigma(L, \Gamma(T_pM/TP))$, where L generates $F[\phi_t^{\#}]\,F^{-1}$, if and only if $\sigma(\phi^{\#}, \Gamma_1)$ lies in an annulus centered at zero and disjoint from the spectrum of $\sigma([\phi^{\#}], \Gamma_2)$. To obtain this we needed the fact that TP and T_pM/TP are vector bundles over P. Now, the result follows from Proposition 5.1 because the existence of the isomorphism F implies that

$$\sigma([L_X], \Gamma_2) = \sigma(L, \Gamma(T_pM/TP). \qquad \text{Q.E.D.}$$

Remark 1. The infinitesimal version of Proposition 5.1 shows that the normal hyperbolicity of ϕ^t at P can be ascertained from an unintegrated condition on the vector field X since one does not need to know the flow in order to compute the spectrum of L_X. For examples of this see [2] in case P is a periodic orbit and [1] when $P = M$ and ϕ^t is the geodesic flow.

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